# Introduction to the SM (1+2)

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# General remarks

- Please ask questions!
- Email: yg73@cornell.edu
- Do your "homework." This is the best way to learn http://lepp.cornell.edu/∼yuvalg/TESHEP/
- The plan:
  - Intro to QFT
  - Intro to the SM
  - A bit about BSM

# What is HEP?

# What is HEP

**Find the basic laws of Nature**

More formally

$$\mathcal{L} = ?$$

- We have quite a good answer
- It is very elegant, it is based on axioms and symmetries
- The generalized coordinates are fields
- We use particles to answer this question

# What is mechanics?

- Answer the question: what is $x(t)$?
- A system can have many DOFs, and then we seek to find $\vec{x}(t) \equiv x_1(t)$, $x_2(t)$,...
- Once we know $\vec{x}(t)$ we know any observable
- Solving for $q_1 \equiv x_1 + x_2$ and $q_2 \equiv x_1 - x_2$ is the same as solving for $x_1$ and $x_2$
- The idea of generalized coordinates is very important

How do we solve mechanics?

# How do we find $x(t)$?

- $x(t)$ minimizes the action, $S$. This is an axiom
- There is one action for the whole system

$$S = \int_{t_1}^{t_2} L(x, \dot{x}) dt$$

- The solution is given by the E-L equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

- Once we know $L$ we can find $x(t)$ up to initial conditions
- Mechanics is reduced to the question "what is $L$?"

# An example: Newtonian mechanics

We assume particle with one DOF and

$$L = \frac{mv^2}{2} - V(x)$$

- We use the E-L equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \qquad L = \frac{mv^2}{2} - V(x)$$

- The solution is $-V'(x) = m\dot{v}$, aka $F = ma$
- Here $F = ma$ is the output, not the starting point!
- Input is $L$. We need to measure the parameters
- So how do we find what is $L$?

# What is L?

$L$ is the most general one that is invariant under some symmetries

- We (again!) rephrase the question. Now we ask what are the symmetries of the system that lead to $L$

# What is field theory

# What is a field?

- In math: something that has a value in each point. We can denote it as $\phi(x,t)$
  - Temperature (scalar field)
  - Wind (vector field)
  - Mechanical string (?)
  - Density of people (?)
  - Electric and magnetic fields (vector fields)
- How good is the field description of each of these?
- In physics a field used to be associated with a source, but now we know that fields are fundamental

# A familiar example: the EM field

- Consider $E(x,t)$. It obeys the wave equation

$$\frac{\partial^2 E(x,t)}{\partial t^2} = c^2 \frac{\partial^2 E(x,t)}{\partial x^2}$$

- The solution is ($\varphi_0$ depends on IC)

$$E(x,t) = A\cos(\omega t - kx + \varphi_0), \qquad \omega = ck$$

- Some important implications of the result
  - Each mode has it own amplitude, $A(\omega)$
  - The energy in each $\omega$ is conserved
  - The superposition principle
- Are the statements above exact?

# How to deal with generic field theories

- $\phi(x,t)$ has an infinite number of DOFs. It can be an approximation for many (but finite) DOFs
- To solve mechanics of fields we need to find $\phi(x,t)$
- Here $\phi$ is the generalized coordinate, while $x$ and $t$ are treated the same (nice!)
- We still need to minimize $S$

$$S = \int \mathcal{L} dx dt \qquad \mathcal{L}[\phi(x,t), \phi'(x,t), \dot{\phi}(x,t)]$$

- We usually require Lorentz invariant (and use $c=1$)

$$S = \int \mathcal{L} d^4x \qquad \mathcal{L}[\phi(x,t), \partial_\mu \phi(x_\mu)]$$

# Solving field theory

We also have an E-L equation for field theories

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \left( \partial_\mu \phi \right)} \right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

- We have a way to solve field theory, just like mechanics. Give me $\mathcal{L}$ and I can know everything!
- Just like in Newtonian mechanics, we want to get $\mathcal{L}$ from symmetries!

# Free field theory

- The "kinetic term" is promoted

$$T \propto \left(\frac{dx}{dt}\right)^2 \Rightarrow T \propto \left(\frac{d\phi}{dt}\right)^2 - \left(\frac{d\phi}{dx}\right)^2 \equiv (\partial_\mu \phi)^2$$

- Free particles, and thus free fields, only have kinetic terms

$$\mathcal{L} = (\partial_\mu \phi)^2 \Rightarrow \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial t^2}$$

- An $\mathcal{L}$ of a free field gives a wave equation
- As in Newtonian mechanics, what used to be the starting point, here is the final result
- Why did we get it?

# Harmonic oscillator

# The harmonic oscillator

Why do we care so much about harmonic oscillators?

- Because we really care about springs?
- Because we really care about pendulums?

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Why do we care so much about harmonic oscillators?

- Because we really care about springs?
- Because we really care about pendulums?

Because almost any function around its minimum can be approximated as a harmonic function!

- Indeed, we usually expand the potential around one of its minima
- We identify a small parameter, and keep only few terms in a Taylor expansion

# Classic harmonic oscillator

$$V = \frac{kx^2}{2}$$

We solve the E-L equation and get

$$x(t) = A\cos(\omega t) \qquad k = m\omega^2$$

- The period does not depend on the amplitude
- Energy is conserved

Which of the above two statements is a result of the approximation of keeping only the harmonic term in the expansion?

# Coupled oscillators

# Coupled oscillators

- There are normal modes
- The normal modes are not “local” as in the case of one oscillator
- The energy of each mode is conserved
- This is an approximation!
- Once we keep non-harmonic terms energy moves between modes

$$V(x,y) = \frac{k_1 x^2}{2} + \frac{k_2 y^2}{2} + \alpha x^2 y$$

- What determines the rate of energy transfer?

# Things to think about

- Relations between harmonic oscillators and free fields

# The quantum SHO

# What is QM?

- Many ways to formulate QM
- For example, we promote $x \to \hat{x}$
- We solve QM if we know the wave function $\psi(x,t)$
- How many wave functions describe a system?
- The wave function is mathematically a field

# The quantum SHO

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \qquad E_n = (n + 1/2)\hbar\omega$$

- We also like to use

$$H = (a^\dagger a + 1/2)\hbar\omega \qquad a, a^\dagger \sim x \pm ip \qquad x \sim a + a^\dagger$$

- We call $a^\dagger$ and $a$ creation and annihilation operators

$$E = a|n\rangle \propto |n-1\rangle \qquad a^\dagger|n\rangle \propto |n+1\rangle$$

- So far this is abstract. What can we do with it?

# Couple oscillators

Consider a system with 2 DOFs and same mass with

$$V(x,y) = \frac{kx^2}{2} + \frac{ky^2}{2} + \alpha xy$$

The normal modes are

$$q_\pm = \frac{1}{\sqrt{2}}(x \pm y) \qquad \omega_\pm^2 = \frac{k \pm \alpha}{m}$$

What is the QM spectrum of this system?

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What is the QM spectrum of this system?

$$(n_+ + 1/2)\hbar\omega_+ + (n_- + 1/2)\hbar\omega_- \qquad |n_+, n_-\rangle$$

# Couple oscillators and Fields

- With many DOFs, $a \to a_i \to a(k)$
- And the states

$$|n\rangle \to |n_i\rangle \to |n(k)\rangle$$

- And the energy

$$(n + 1/2)\hbar\omega \to \sum (n_i + 1/2)\hbar\omega_i \to \int [n(k) + 1/2]\hbar\omega dk$$

- Just like in mechanics, we expand around the minimum of the fields, and to leading order we have SHOs
- In QFT fields are operators while $x$ and $t$ are not

# SHO and photons

I have two questions:

- What is the energy that it takes to excite a SHO by one level?
- What is the energy of the photon?

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- What is the energy that it takes to excite a SHO by one level?
- What is the energy of the photon?

Same answer

$$\hbar\omega$$

- Why the answer to both question is the same? Can we learn anything from it?

# What is a particle?

Excitations of SHOs are particles

# More on QFT

# What about masses?

- A “free” Lagrangian gives massless particle

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 \Rightarrow \omega = k \quad (\text{or } E = P)$$

- We can add “potential” terms (without derivatives)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2$$

- Here $m$ is the mass of the particle. Still free particle
- (HW) Show that $m$ is a mass of the particle by showing that $\omega^2 = k^2 + m^2$

# What about other terms?

- How do we choose what terms to add to $\mathcal{L}$?
- Must be invariant under the symmetries
- We keep some leading terms (usually, up to $\phi^4$)
- Lets add $\lambda\phi^4$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4$$

- We get the non-linear wave equation

$$\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial t^2} = m^2\phi + \lambda\phi^3$$

- We do not know how to solve it

# What about fermions?

- We saw how massless and massive bosons are related to SHO
- The spin of the particle is related to the polarization of the classical field
  - Scalar field : spin zero paricles
  - vector field : spin one paricles
- We can construct a fermion SHO

$$[a, a^\dagger] = 1 \quad \to \quad \{b, b^\dagger\} = 1$$

- No classical analogue since $b^2 = 0$
- We can then think of fermionic fields. They can generate only one particle in a given state

# A short summary

- Particles are excitations of fields
- The fundeumental Lagrangian is giving in term of fields
- Our aim is to find $\mathcal{L}$
- We can only solve the linear case, that is, the equivalent of the SHO
- What can we do with higher order temrs?

# Perturbation theory

# Perturbation theory

$$H = H_0 + H_1 \qquad H_1 \ll H_0$$

- In many cases pertubation theory is a mathematical tool
- There are cases, however, that PT is a better way to describe the physics
- Many times we prefer to work with EV of $H$ (why?)
- Yet, at times it is better to work with EV of $H_0$ (why?)

# 1st and 2nd order PT

$$H = H_0 + H_1 \qquad H_1 \ll H_0$$

- In first order we care only about the states with the same energy

$$\langle f|H_1|i\rangle \qquad E_f = E_i$$

- 2nd order pertubation theory probe the whole spectrum

$$\sum_n \frac{\langle f|H_1|n\rangle\langle n|H_1|i\rangle}{E_n - E_f} \qquad E_f = E_i \qquad E_n \neq E_i$$

# PT for 2 SHOs

$$V(x,y) = \frac{kx^2}{2} + \frac{4ky^2}{2} + \alpha x^2 y$$

- Classically $\alpha$ moves energy between the two modes
- How it goes in QM?
- Recall the Fermi golden rule

$$P \propto |\mathcal{A}|^2 \times \mathrm{P.S.} \qquad \mathcal{A} \sim \langle f|\alpha x^2 y|i\rangle$$

- The relevant thing to calculate is the transition amplitude, $\mathcal{A}$.

# Transitions

$$V(x,y) = \frac{kx^2}{2} + \frac{4ky^2}{2} + \alpha x^2 y$$

- Recall

$$x \sim a_x + a_x^\dagger \qquad y \sim a_y + a_y^\dagger$$

- When $\mathcal{A}$ is non-zero?

$$\mathcal{A} \sim \langle f | \alpha x^2 y | i \rangle$$

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- Since $H_1 \sim x^2 y$ we see that $\Delta n_y = \pm 1$ and $\Delta n_x = 0, \pm 2$
- What could you say if the perturbation was $x^2 y^3$?

# Two SHOs with small $\alpha$

$$V(x,y) = \frac{kx^2}{2} + \frac{4ky^2}{2} + \alpha x^2 y \qquad \omega_y = 2\omega_x$$

- Consider $|i\rangle = |0,1\rangle$
- Since $\omega_y = 2\omega_x$ only $f = |2,0\rangle$ is allowed by energy conservation and by the perturbation

$$\mathcal{A} \sim \langle 2,0|\alpha x^2 y|0,1\rangle \sim \alpha\langle 2,0|(a_x+a_x^\dagger)(a_x+a_x^\dagger)(a_y+a_y^\dagger)|0,1\rangle$$

- $a_y$ in $y$ annihilates the $y$ "particle" and $(a_x^\dagger)^2$ in $x^2$ creates two $x$ "particles"
- It is a decay of a particle $y$ into two $x$ particles with width $\Gamma \propto \alpha^2$ and thus $\tau = 1/\Gamma$

# Even More PT

$$H_1 = \alpha x^2 z + \beta xyz \qquad \omega_z = 10,\ \omega_y = 3,\ \omega_x = 1$$

- Calculate $y \to 3x$ using 2nd order PT

$$\mathcal{A} \sim \langle 3,0,0|\mathcal{O}|0,1,0\rangle \qquad \mathcal{O} \sim \sum \frac{\langle 3,0,0|V'|n\rangle\langle n|V'|0,1,0\rangle}{E_n - E_{0,1,0}}$$

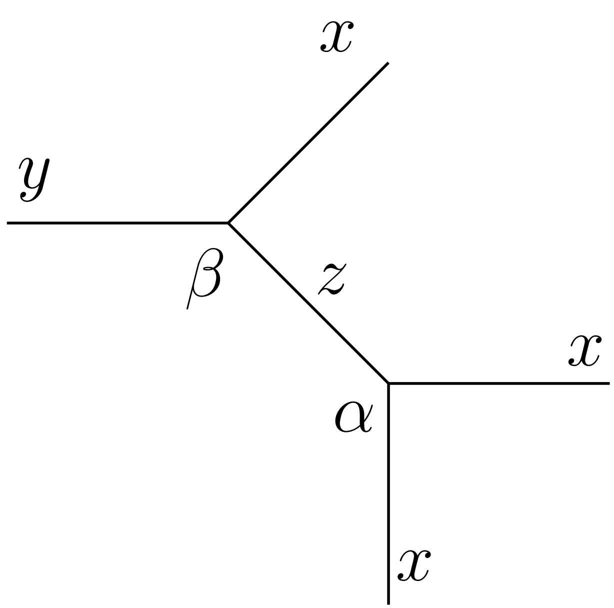
- Which intermediate states? $|1,0,1\rangle$ and $|2,1,1\rangle$
  - $\mathcal{A}_1 = |0,1,0\rangle \xrightarrow{\beta} |1,0,1\rangle \xrightarrow{\alpha} |3,0,0\rangle$
  - $\mathcal{A}_2 = |0,1,0\rangle \xrightarrow{\alpha} |2,1,1\rangle \xrightarrow{\beta} |3,0,0\rangle$
- The total amplitude is then

$$\mathcal{A} \propto \alpha\beta \left(\frac{\#}{\Delta E_1} + \frac{\#}{\Delta E_2}\right) \propto \alpha\beta \left(\frac{\#}{8} + \frac{\#}{12}\right)$$

# Closer look

$$V' = \alpha x^2 z + \beta xyz \qquad \omega_z = 10,\ \omega_y = 3,\ \omega_x = 1$$

- We look at $\mathcal{A} = |0,1,0\rangle \xrightarrow{\beta} |1,0,1\rangle \xrightarrow{\alpha} |3,0,0\rangle$




$$\mathcal{A} \propto \frac{\alpha\beta}{\Delta E_z}$$

# Feynman diagrams

# Using PT for fields

- For SHOs we have $x_i \sim a_i + a_i^\dagger$
- For fields we then have

$$\phi \sim \int \left[a(k) + a^\dagger(k)\right] dk$$

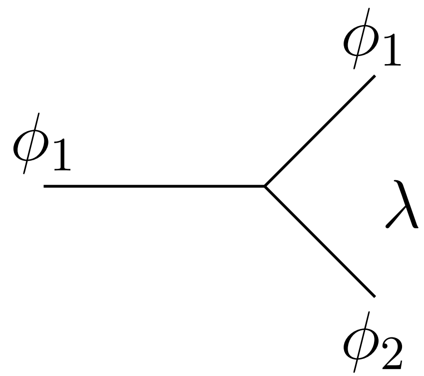
**Quantum field $=$ creation and annihilation operators**

# Feynman diagrams

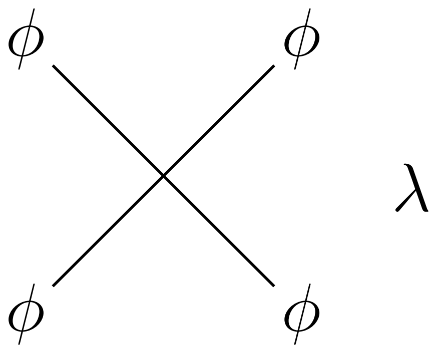
- A graphical way to do perturbation theory with fields
- Unlike SHOs before, a particle can have any energy as long as $E \geq m$
- Operators with 3 or more fields generate transitions between states. They give decays and scatterings
- Decay rates and scattering cross sections are calculated using the Golden Rule
- Amplitudes are calculated from $\mathcal{L}$
- We generate graphs where lines are particles and vertices are interactions

# Examples of vertices

$\mathcal{L} = \lambda \phi_1^2 \phi_2$ :




$\mathcal{L} = \lambda \phi^4$ :



# Calculations

- We usually care about $1 \to n$ or $2 \to n$ processes
- We need to make sure we have energy conservation
- External (Internal) particles are called on(off)–shell
  - On-shell: $E^2 = p^2 + m^2$
  - Off-shell: $E^2 \neq p^2 + m^2$
- $\mathcal{A} =$ the product of all the vertices and internal lines
- Each internal line with $q^\mu$ gives suppression

$$\frac{1}{m^2 - q^2}$$

- There are many more rules to get all the factors right

# Examples of amplitudes

$$\mathcal{L} = \lambda_1 XYZ + \lambda_2 X^2 Z$$

$$\Gamma(Z \to XY)$$

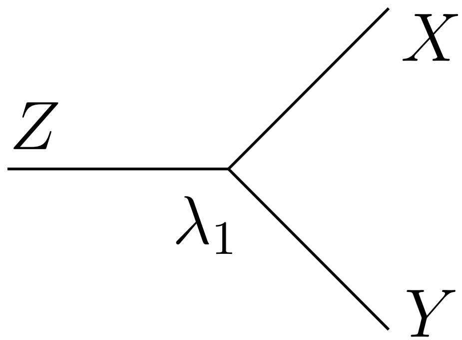
- Energy conservation condition
- Draw the diagram and estimate the amplitude

# Examples of amplitudes

$$\mathcal{L} = \lambda_1 XYZ + \lambda_2 X^2 Z$$

$$\Gamma(Z \to XY)$$

- Energy conservation condition $m_Z > m_X + m_Y$
- Draw the diagram and estimate the amplitude




$$\mathcal{A} \propto \lambda_1$$

# Examples of amplitudes (2)

$$\mathcal{L} = \lambda_1 XYZ + \lambda_2 X^2 Z$$

$$\boxed{\Gamma(Y \to 3X)}$$

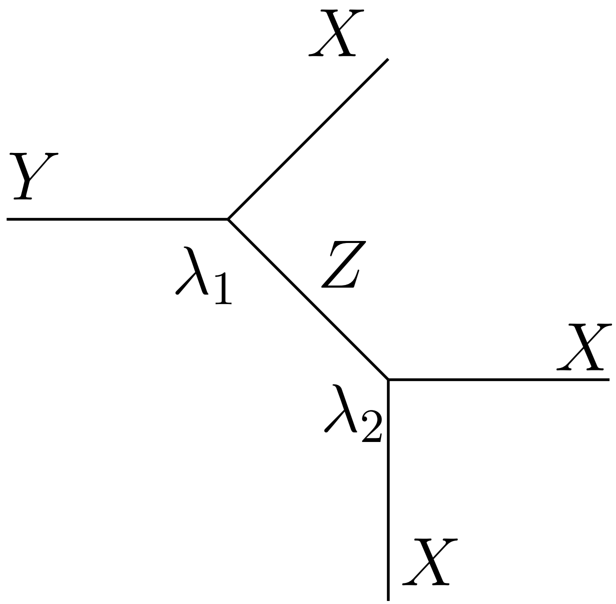
- Energy conservation condition
- Draw the diagram and estimate the amplitude

# Examples of amplitudes (2)

$$\mathcal{L} = \lambda_1 XYZ + \lambda_2 X^2 Z$$

$$\boxed{\Gamma(Y \to 3X)}$$

- Energy conservation condition $m_Y > 3m_X$
- Draw the diagram and estimate the amplitude




$$\mathcal{A} \propto \lambda_1 \lambda_2 \times \frac{1}{\Delta E_Z^2}$$

$$= \lambda_1 \lambda_2 \times \frac{1}{m_Z^2 - q^2}$$

# Examples of amplitudes (HW)

$$\mathcal{L} = \lambda_1 XYZ + \lambda_2 X^2 Z$$

$$\sigma(XX \to XY)$$

- Energy conservation condition
- Draw the diagram and estimate the amplitude

# Some summary

- Quadratic terms describe free fields. Free particles cannot be created nor decay
- We use perturbation theory were terms with 3 or more fields in $\mathcal{L}$ are consider small
- These terms can generate and destroy particles and give dynamics
- Feynman diagrams are a tool to calculate transition amplitudes
- Many more details are needed to get calculation done
- Once calculations and experiments to check them are done, we can test our theory