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Cornell University, Department of Physics

TES-HEP summer school 2015, HW # 1.

Question 1: E-L for Fields

The E-L for fields is given by

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} \right) = \frac{\partial \mathcal{L}}{\partial \phi} \tag{1}$$

And we can write it explicitly for the case of $1+1$ dimentions

$$\frac{\partial}{\partial t} \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_t \phi\right)} \right) - \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_x \phi\right)} \right) = \frac{\partial \mathcal{L}}{\partial \phi} \tag{2}$$

1. Show that $\mathcal{L}$ for a free field gives the free wave equation, that is

$$\mathcal{L} = \left(\partial_\mu \phi\right)^2 = \left(\partial_t \phi\right)^2 - \left(\partial_x \phi\right)^2 \quad \Longrightarrow \quad \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \phi}{\partial t^2} \tag{3}$$

2. Assume a solution of the form $\phi(x,t) = \cos(\omega t + kx)$ and show that it is indeed a solution with $\omega = k$.

3. We now add a term to $\mathcal{L}$ of the form $m^2\phi^2$. Find the modified wave equation, and assuming the same solution, show that $\omega^2 = k^2 + m^2$. Explain why after quantization that correspond to a particle of mass m.

Question 2: Lagrangian formalism of E&M

In mechanics, one can start from $L = ma$ or it can be derived form $L = mv^2/2 - V(x)$ based on the principle of minimal action. In this problem you will show that Maxwell's equations can be derived for a Lagrangian density and the principle of minimal action.

We start with some definitions. We define the vector potential and the electromagnetic tensor

$$A_\mu(x_\mu) = \left\{\phi(x_\mu), A_i(x_\mu)\right\}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4}$$

$$\mu, \nu = 0, 1, 2, 3; \qquad i, j, k = 1, 2, 3. \tag{5}$$

The electric field (E) and magnetic field (B) are then

$$E_i = F^{i0}, \qquad B_i = -\frac{1}{2}\epsilon_{ijk} F^{jk}, \tag{6}$$

where the ϵ tensor is defined by $\epsilon_{123} = 1$ and that it is antisymmetric under the exchange of any two indices. Explicitly, $F_{\mu\nu}$ is given by:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{7}$$

We further define the dual field strength tensor

$$\tilde{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}F_{\mu\nu}, \tag{8}$$

where $\epsilon^{\alpha\beta\mu\nu}$ is the completely antisymmetric rank 4 epsilon tensor. Clearly F is antisymmetric. Moreover, you can show (check it!) that

$$F_{\mu\nu}F^{\mu\nu} = 2\left(B^2 - E^2\right), \qquad \tilde{F}_{\mu\nu}F^{\mu\nu} = -4\left(B \cdot E\right). \tag{9}$$

The above two combinations are manifestly Lorentz invariant. We also define the four current

$$J^\alpha = (\rho, J_i). \tag{10}$$

We consider Maxwell's equations

$$\nabla \cdot E = \rho, \qquad \nabla \times B - \frac{\partial E}{\partial t} = J_i, \qquad \nabla \cdot B = 0, \qquad \nabla \times E + \frac{\partial B}{\partial t} = 0. \tag{11}$$

1. Show that the inhomogeneous equations (that is, the first two) can be written as

$$\partial_\alpha F^{\alpha\beta} = J^\beta. \tag{12}$$

2. We assume the following Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + J^\mu A_\mu, \tag{13}$$

and consider each of A_μ to be an independent DOF. Using the E-L equation show that this Lagrangian gives the two inhomogeneous equations.

Question 3: Fermionic harmonic oscillator

In this question we study the simple Fermionic harmonic oscillator. You must be familiar with the bosonic SHO (usually just called the SHO), which is defined with

$$H = \hbar\omega\frac{a^\dagger a + aa^\dagger}{2} = \hbar\omega(N + 1/2), \qquad N = a^\dagger a, \tag{14}$$

with

$$[a, a^\dagger] = 1, \qquad [a, a] = [a^\dagger, a^\dagger] = 0. \tag{15}$$

Here we study a similar system that describes a Fermionic oscillator, which has no classical analog. Instead of using commutation relations for the raising and lowering operators, we define fermionic versions with anti-commutation relations:

$$\{b, b^\dagger\} = 1, \qquad \{b, b\} = \{b^\dagger, b^\dagger\} = 0, \tag{16}$$

where $\{a, b\} = ab + ba$. For the Hamiltonian we have

$$H = \hbar\omega \frac{b^\dagger b - b b^\dagger}{2} = (N - 1/2), \qquad N = b^\dagger b. \tag{17}$$

1. Show that $b^2 = 0$.

2. Show that there is a state, which we denote by $|0\rangle$, that is annihilated by b.

3. We define $|1\rangle = b^\dagger|0\rangle$. Show that this state cannot vanish.

4. We assume that all possible operators must be composed of products of b and $b^\dagger$. Show that $|1\rangle$ and $|0\rangle$ span the Hilbert space. (One way to go is to show that there are a total of four independent operators. There is a theorem that states that this number is equal to the dimension of the Hilbert space squared.)

5. What is the energy of each state?

6. Since we have a two level system lets us write explicitly

   $$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{18}$$

   In this basis write b, $b^\dagger$ and N. Write them also in terms of the Pauli matrices and the unit matrix.

7. Consider a system with one harmonic and one bosonic oscillator with the same ω. What is the vacuum energy for that system?