Question 1: The quadrupole

We discuss some properties of the quadrupole and see some examples.

1. Show that
   \[ T_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} \left( \frac{1}{r} \right) = \frac{(3\hat{r}_i \hat{r}_j - \delta_{ij})}{r^3} \]
   and write it explicitly as a $3 \times 3$ matrix.

2. We now consider a set of 4 point charges, 2 of charge $+q$ and 2 of charge $-q$, arranged in corners of a square, as shown in Fig. 2-6, p. 56 of Heald and Marion. Choose a coordinate system with the origin at the center of the square, as in the figure. Show that the potential of this charge configuration at a distance $r \gg L$ from the origin has vanishing monopole and dipole moments and that the leading order term is the quadrupole, and write $Q_{ij}$ explicitly as a $3 \times 3$ matrix.

Answer: We write
   \[ \Phi(r) \approx \frac{1}{6} Q_{ij} (3x_i x_j - r^2 \delta_{ij}) = \frac{1}{6} Q_{ij} \left( \frac{3\hat{r}_i \hat{r}_j - \delta_{ij}}{r^3} \right), \]
   where $i, j = 1, 2, 3$ and $x_i$’s are the components of $r$ in a rectangular coordinate system. Using this normalization
   \[ Q_{ij} = 3 \sum_{\alpha} q_{\alpha} x'_{\alpha,i} x'_{\alpha,j} \]
   (Some of you may use different normalization and/or subtract the trace, for example
   \[ Q_{ij} = \sum_{\alpha} q_{\alpha} x'_{\alpha,i} x'_{\alpha,j}, \quad \text{or} \quad Q_{ij} = 3 \sum_{\alpha} q_{\alpha} (x'_{\alpha,i} x'_{\alpha,j} - r^2 \delta_{ij}) \]
   All of these are OK as long as you remember to use the correct formula for the potential.
Let us find the

Monopole: \[ \sum_{\alpha=1}^{4} q_\alpha = q + q - q - q = 0 \]

Dipole: \[ \sum_{\alpha=1}^{4} q_\alpha x'_{\alpha,1} = +q \frac{L}{2} + q \left( -\frac{L}{2} \right) = 0 \]
\[ \sum_{\alpha=1}^{4} q_\alpha x'_{\alpha,2} = (-q) \frac{L}{2} + q \left( -\frac{L}{2} \right) = 0 \]
\[ \sum_{\alpha=1}^{4} q_\alpha x'_{\alpha,3} = 0 \]

Quadrupole: \[ 3 \sum_{\alpha=1}^{4} q_\alpha x'_{\alpha,1} x'_{\alpha,1} = +3q \left( \frac{L}{2} \right)^2 + 3q \left( -\frac{L}{2} \right)^2 = 3qL^2 \]
\[ 3 \sum_{\alpha=1}^{4} q_\alpha x'_{\alpha,2} x'_{\alpha,2} = 0 \]
\[ 3 \sum_{\alpha=1}^{4} q_\alpha x'_{\alpha,1} x'_{\alpha,3} = 0 \]
\[ 3 \sum_{\alpha=1}^{4} q_\alpha x'_{\alpha,3} x'_{\alpha,1} = 0 \]
\[ 3 \sum_{\alpha=1}^{4} q_\alpha x'_{\alpha,2} x'_{\alpha,2} = 0 \]
\[ 3 \sum_{\alpha=1}^{4} q_\alpha x'_{\alpha,1} x'_{\alpha,3} = 0 \]
\[ 3 \sum_{\alpha=1}^{4} q_\alpha x'_{\alpha,3} x'_{\alpha,3} = 0 \]

We see that the largest surviving term is the quadrupole, with

\[ Q_{ij} = \frac{3qL^2}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]
\[ \Phi \approx \frac{3qL^2}{4} \frac{x^2_1 - x^2_2 + r^2}{r^5} = \frac{3qL^2}{4} \frac{x^2_1 - x^2_2}{r^5} \]
It is conventional to remove the trace of the matrix by subtracting the matrix $Q$ by $Q_{ii}1$, but in this case, the trace is zero, so it doesn’t matter.

3. Write $\Phi$ is spherical coordinate, that is, $\Phi(r, \theta, \phi)$  

**Answer:** In spherical coordinates

$$\Phi = \frac{3qL^2}{4} \frac{r^2 \sin^2 \theta \cos^2 \phi - r^2 \sin^2 \theta \sin^2 \phi}{r^5} = \frac{3qL^2 \sin^2 \theta \cos(2\phi)}{4} \frac{1}{r^3}$$  (7)

4. Find the approximate expression (again at $r \gg L$) for the electric field $\vec{E}$ of this charge configuration, in spherical coordinates. (See H&M Eq. (A.49) for the expression of the gradient in spherical coordinates.)  

**Answer:** Now we take the gradient in polar coordinates to find $\vec{E}$.

$$E_r = -\frac{\partial \Phi}{\partial r} = \frac{9qL^2 \sin^2 \theta \cos(2\phi)}{4} \frac{1}{r^4}$$  

$$E_\theta = -\frac{1}{r} \frac{r}{\partial \theta} \frac{\partial \Phi}{\partial \theta} = -\frac{3qL^2 \sin \theta \cos \theta \cos(2\phi)}{2} \frac{1}{r^4}$$  (8)

$$E_\phi = -\frac{1}{r \sin \theta} \frac{\partial \Phi}{\partial \phi} = \frac{3qL^2 \sin \theta \sin(2\phi)}{2} \frac{1}{r^4}$$

5. We now consider a similar setup but where all the charges are positive. Find the monopole, dipole and quadropole moments for that setup.

**Answer:** The total charge is $Q = 4q$. The dipole vanishes, and the quadrople is

$$Q_{ij} = \frac{3qL^2}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$  (9)

or if we subtract the trace

$$Q_{ij} = \frac{qL^2}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$  (10)

**Question 2:** More on Multipoles
Consider a charge distribution as shown in the picture below. \((q_0\) in the middle, \(q_1 + q_2\) at \(a\) and \(-q_1 + q_2\) at \(-a\) on the \(z\) axis, and \(-q_2\) at \(±a\) on the \(x\) axis.) We define \(d = q_1a\) and \(m = q_2a^2\).

1. We first consider the situation where \(q_2 = 0\) and \(q_1 ≫ q_0\). For generic values of \(θ\) and \(ϕ\), there is a crossover radius \(r_C\) such that as we vary \(r\), the potential \(Φ\) becomes dominated by the monopole term once \(r\) crosses \(r_C\). Give a rough estimate of \(r_C\) in terms of the parameters given in the question, and state whether the monopole dominates when \(r\) is above or below \(r_C\).

Answer:
When \(q_2 = 0\), we basically have a monopole \(q_0\) and a dipole in the \(z\)-direction with dipole moment \(\vec{p} = 2q_1a\hat{z} = 2\hat{d}\). The combined potential is then

\[
Φ = \frac{q_0}{r} + \frac{2d \cos \theta}{r^2}
\]

Roughly speaking, ignoring factors of 2 and so on, the monopole dominates when

\[
\frac{q_0}{r} > \frac{d}{r^2}
\]

\[
⇒ r > \frac{d}{q_0} = a\frac{q_1}{q_0} ≡ r_C
\]

**Common mistakes:**
Including a \(\cos θ\) dependence, since we are only asking for a typical scale using the parameters of the question.

2. Still consider the same situation where \(q_2 = 0\) and \(q_1 ≫ q_0\). Now we discuss the electric field \(\vec{E}\). Are there specific values of \(θ\) where the monopole term is always the most important one, regardless of whether \(r\) is above or below \(r_C\)? For these values,
specifically indicate for which component of \( \vec{E} \) (that is \( E_r \), \( E_\theta \) and \( E_\phi \)) this is the case.

**Answer:**
Since the monopole electric field is radial, this can only occur for the radial components of \( \vec{E} \). The radial component of the electric field is given by taking \(-\frac{\partial}{\partial r}\) of the potential

\[
E_r = \frac{q_0}{r^2} + \frac{4d \cos \theta}{r^3} \tag{13}
\]

The monopole term can dominate even for very small \( r \) if the angle \( \theta \) is such that \( \cos \theta \) is close to zero, i.e. when \( \theta \approx 90^\circ \).

**Common mistakes:**
Some of you noticed that the dipole contribution to \( E_\theta \) vanishes at \( \theta = 0^\circ \) and \( 180^\circ \), so you thought that the monopole contribution to \( E_\theta \) will dominate at these angles. However, the monopole doesn’t contribute to \( E_\theta \) at all!

3. We now consider the \( q_2 \neq 0 \) case and assume that \( q_0 \sim q_2 \) and \( q_1 \gg q_0 \). Explain why there is no range of \( r \) values where the quadrupole term is generically the dominant contribution to the potential \( \Phi \). Remember that we only consider values of \( r \gg a \).

**Answer:**
Roughly speaking, keeping only the radial dependence, we have

\[
\Phi \sim \frac{q_0}{r} + \frac{d}{r^2} + \frac{m}{r^3} \tag{14}
\]

The quadrupole dominates the monopole when \( r^2 \lesssim m/q_0 = a^2q_2/q_0 \sim a^2 \), and the dipole when \( r \lesssim m/d = aq_2/q_1 \ll a \). Since we are considering \( r \gg a \), it is impossible to dominate the monopole nor dipole term.

**Common mistakes:**
Incomplete arguments. E.g. some of you said that since the quadrupole term has a higher \( \frac{1}{r^2} \) dependence, and \( r \gg a \), hence it cannot dominate. This is incomplete because you also have to discuss the strength of the quadrupole (i.e. point out that \( q_2 \sim q_0 \ll q_1 \)). For instance, back in Part 1, the dipole still dominated the monopole for \( a \ll r < r_C \) despite a higher \( \frac{1}{r} \) dependence, because we had \( q_1 \gg q_0 \). The same could have happened for the quadrupole had we considered a different scenario with \( q_2 \gg q_1 \).

4. Still, within the same situation (\( q_0 \sim q_2 \) and \( q_1 \gg q_0 \)), even though the quadrupole term is never the dominant term in the potential, it still cannot be neglected for a generic point with \( r \gg a \), since there are some physical observables which only the quadrupole potential can generate. Explain why it indeed cannot be neglected by identifying such an observable.
Answer:
The quadrupole potential is the only one that generate a finite $E_\varphi$ component, since neither the monopole or dipole terms contribute to it.

**Common mistakes:**
We specifically asked for a physical observable, but some of you only mentioned that the quadrupole term breaks azimuthal symmetry, without mentioning an observable.