Question 1: Fermi’s golden rule

Read in any book or website about the Fermi golden rule. Very briefly explain when it is applied and what assumptions must be met to use it.

Answer: Actually many different derivations of Fermi’s golden rule can be found in literature, which may potentially confuse a first-time learner. Some consider a time-dependent perturbation such as a sinusoidal driving field (Griffiths; Gasiorowicz). Others consider a particle scattering off a static potential (Gottfried & Yan). Some argue that the sinusoidal driving field has a finite bandwidth (Griffiths). Others argue that energy levels are actually broadened (the S16 grader’s lecturer Prof. Andrew Steane back in 2003 at Oxford). Yet others argue that one should consider a continuum of final states (Gottfried and Yan; Gasiorowicz). Finally, mathematical physicists even claim that a Hamiltonian system cannot exhibit true Golden rule behaviour at very large times due to a phenomenon called re-scattering, and that you really need to consider an open quantum system that includes interaction with its surroundings (Rep. Prog. Phys., 41, 587).

Whichever version you subscribe to, there are usually two parts to the derivation:

1. The transition probability at a specific energy is proportional to $|\langle f | V | i \rangle|^2 \frac{\sin^2(\Delta E t)}{\Delta E^2}$, where $\langle f | V | i \rangle$ is the matrix element of the perturbation between the initial and final states, and $\Delta E = E_f - E_i$ (or $E_f - E_i - \hbar \omega$ if you are considering a time-dependent sinusoidal perturbation.)

2. You actually need to integrate this over $\Delta E$ multiplied by some weight function $w(\Delta E)$, as a result of linewidth, continuum of states, etc.

Notice that the function $f(\Delta E) \equiv \frac{\sin^2(\Delta E t)}{(\Delta E)^2}$ becomes increasingly narrow in $\Delta E$ as $t$ increases. In particular, the width falls as $1/t$ and the height grows as $t^2$. Let’s consider what happens when you integrate $f(\Delta E)w(\Delta E)$ over $\Delta E$.

1. At small $t$, it is possible that the weight function $w(\Delta E)$ is more narrow than $f(\Delta E)$ (especially if you are dealing with discrete states). In that case, when you consider the integral, the height grows as $t^2$ while the width is set by $w(\Delta E)$ and hence constant. As a result, the transition probability grows quadratically as $t^2$. This is indeed the small $t$ behaviour of Rabi oscillations.
2. However, if \( f(\Delta E) \) is more narrow than \( w(\Delta E) \), then when you consider the integral, the height grows as \( t^2 \) while the width falls as \( 1/t \), so the transition probability grows only linearly as \( t \). Therefore, if you consider the rate of change of the transition probability, it is a constant related to \(|\langle f|V|i\rangle|^2\). This is the main results of Fermi’s golden rule, which incidentally was invented actually by Dirac, not Fermi.

Therefore, Fermi’s golden rule is valid when we are in the regime where the transition probability grows linearly as \( t \). This naturally occurs if we are dealing with a continuum set of final states. If we are considering discrete final states, it only becomes valid at times large compared to the linewidth.

**Question 2: Harmonic oscillator perturbation theory**

Consider a system with 3 DOFs with the following Hamiltonian
\[
H = \frac{p_x^2}{2m} + \frac{m\omega_x^2 x^2}{2} + \frac{p_y^2}{2m} + \frac{m\omega_y^2 y^2}{2} + \frac{p_z^2}{2m} + \frac{m\omega_z^2 z^2}{2} + \lambda_1 xyz + \lambda_2 x^2 z.
\]  
(1)

We further assume that \( \omega_y = 3\omega_x, \omega_z \gg \omega_y \) and that \( \lambda_1 \) and \( \lambda_2 \) are small and thus can be treated as perturbation. We denote a state of the system as \(|n_x, n_y, n_z\rangle\). In this question you are asked to use two ways to calculate the transition matrix element
\[
\mathcal{A}(|0, 1, 0\rangle \rightarrow |3, 0, 0\rangle).
\]  
(2)

1. Use second order perturbation theory to show that
\[
\mathcal{A} = c \times \frac{\lambda_1 \lambda_2}{\omega_z^2 - (2\omega_x)^2}
\]  
and find what is \( c \).

As a reminder, if there is no first-order transition, the transition is second-order with matrix elements given by
\[
\mathcal{A}(|I\rangle \rightarrow |F\rangle) = \sum_a \frac{\langle F|H_I|a\rangle \langle a|H_I|I\rangle}{E_I - E_a}
\]  
(4)

where \(|I\rangle \) and \(|F\rangle \) are initial and final states, \( H_I \) is the perturbation, and we are summing over all possible intermediate states \(|a\rangle\). Obviously, there are an infinite number of intermediate states, so stare hard at \( H_I \) to determine which are the intermediate states such that \( \langle F|H_I|a\rangle \langle a|H_I|I\rangle \) is nonzero. (Hint: There are only two.)

**Answer:** There are two intermediate states that contributes, and we have
\[
\mathcal{A} = \lambda_1 \lambda_2 \left[ \frac{\langle 3, 0, 0|x^2z|1, 0, 1\rangle \langle 1, 0, 1|xyz|0, 1, 0\rangle}{-(E_z - 2E_x)} + \frac{\langle 3, 0, 0|xyz|2, 1, 1\rangle \langle 2, 1, 1|x^2z|0, 1, 0\rangle}{-(E_z + 2E_x)} \right]
\]  
(5)
where we used $E_y = 3E_x$. We then use

$$\hat{q} = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger)$$

and obtain

$$\langle 1, 0, 1 | xyz | 0, 1, 0 \rangle = c_1, \quad \langle 3, 0, 0 | x^2z | 1, 0, 1 \rangle = \sqrt{6}c_2,$$
$$\langle 3, 0, 0 | xyz | 2, 1, 1 \rangle = \sqrt{3}c_1, \quad \langle 2, 1, 1 | x^2z | 0, 1, 0 \rangle = \sqrt{2}c_2. \tag{7}$$

with

$$c_1 = \sqrt{\frac{\hbar^3}{2^3 m^3 \omega_x \omega_y \omega_z}}, \quad c_2 = \sqrt{\frac{\hbar^3}{2^3 m^3 \omega_x^2 \omega_y}}, \quad c_1c_2 = \frac{\hbar^3}{2^3 m^3 \omega_x \omega_y^{3/2} \omega_z^{1/2}} \tag{8}$$

We then obtain

$$A = -\frac{\lambda_1 \lambda_2 \sqrt{6}\hbar^3}{2^3 m^3 \omega_x \omega_y^{3/2} \omega_z^{1/2}} \times \left[ \frac{1}{E_z - 2E_x} + \frac{1}{E_z + 2E_x} \right] = -\frac{\lambda_1 \lambda_2 \sqrt{6}\hbar^2}{2^2 m^3 \omega_x^{3/2} \omega_y^{1/2}} \times \left[ \frac{1}{\omega_z^2 - (2\omega_x)^2} \right] \tag{9}$$

So we conclude that

$$c = -\frac{\sqrt{6}\hbar^2}{2^2 m^3 \omega_x^{3/2} \omega_y^{1/2}} = -\frac{\sqrt{2}\hbar^2}{2^2 m^3 \omega_x^2} \tag{10}$$

and for the amplitude we get

$$A = -\frac{\lambda_1 \lambda_2 \sqrt{2}\hbar^2}{4 m^3 \omega_x^2 [\omega_z^2 - (2\omega_x)^2]} \tag{11}$$

2. Use the Feynman diagram for Harmonic oscillator method to get the same result. For that, first draw the diagram and calculate it using the $\hbar = m = 1$ units. The amplitude is the product of the following factors

(a) For each vertex multiply the amplitude by the corresponding coupling constant.

(b) For each propagator use $-1/(E_z^2 - q^2)$ where $E_z$ is the energy of the internal state and $q$ is the energy that go out of it.

(c) Use the correct normalization: (i) For each external state use $1/\sqrt{2\omega_i}$ and (ii) for each final state that appear $n$ times multiply the amplitude by $\sqrt{n}$! (this factor is called symmetry factor).

Compare this result to the one you got in the first item.

Answer:
From Fig. (2) we get

\[ \lambda_1 \lambda_2 \times \frac{-1}{E_z^2 - (2E_x)^2} \times \sqrt{6} \times \frac{1}{4\sqrt{E_y E_x}} = \frac{-\lambda_1 \lambda_2}{E_z^2 - (2E_x)^2} \sqrt{2} \times \frac{1}{4E_z^2} \]  

(12)

Which is the same as what we got before (setting \( \hbar = m = 1 \)) in Eq. (11).

**Question 3: More Harmonic oscillators**

Consider a case with four oscillators, called \( w, x, y, z \) and assume that \( y \to 2x2w \) is allowed by energy conservation. We use the following interaction

\[ \lambda_1 yz^2 + \lambda_2 zx^2 + \lambda_3 zw^2. \]  

(13)

1. Draw the diagram and calculate the transition matrix element using the Feynman rules and the correct normalization.

   **Answer:**

   \[ \frac{2\lambda_1 \lambda_2 \lambda_3}{\sqrt{2}\omega_y(2\omega_x)(2\omega_z)} \times \frac{1}{\omega_z^2 - (2\omega_x)^2} \times \frac{1}{\omega_z^2 - (2\omega_w)^2} \]  

(14)

2. (Optional) Calculate the amplitude using perturbation theory. For that find the 6 intermediate states that contribute and calculate the 6 matrix elements and add them up. Verify that the result of the two methods agree.
FIG. 2: Feynman diagram for $y \rightarrow 2x2w$, with labels showing the various contributions to the calculation.